Some new class of Chaplygin Wormholes

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Abstract

Some new class of Chaplygin wormholes are investigated in the framework of a Chaplygin gas with equation of state $p = -\frac{A}{\rho}$, $A > 0$. Since empty spacetime ($p = \rho = 0$) does not follow Chaplygin gas, so the interior Chaplygin wormhole solutions will never asymptotically flat. For this reason, we have to match our interior wormhole solution with an exterior vacuum solution i.e. Schwarzschild solution at some junction interface, say $r = a$. We also discuss the total amount of matter characterized by Chaplygin gas that supplies fuel to construct a wormhole.

Introduction: Observations of Type IA supernovae and Cosmic microwave background anisotropy suggest that the Universe is currently undergoing an accelerated expansion [1-3]. After the publication of these observational reports, theoretical physicists have been starting to explain this astonishing phenomena theoretically. It is readily understand that this current cosmological state of the Universe requires exotic matter that produces large negative pressure. It is known as dark energy and it fails to obey Null Energy Condition (NEC). To describe theoretically this ghost like matter source, Cosmologists have been proposed several propositions as Cosmological constant [4](the simplest and most popular candidate), Quintessence [5](a slowly evolving dynamical quantity which has a spatially inhomogeneous component of energy with negative pressure), Dissipative matter fluid [6], Chaplygin gas[7] (with generalized as well as modified forms), Phantom energy [8-10](here, the equation of state of the form as $p = -w\rho$ with $w > 1$), Tracker field [11-12](a new form of quintessence) etc. Recently, scientific community shows great interest in wormhole physics because this opens a possibility to trip a very large distance in a very short time. In a pioneering work, Morris and Thorne [13] have shown that wormhole geometry could be found from Einstein general theory of relativity. But one has to tolerate the violation of NEC.

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That means the requirement of matter sources are the same as to explain the recent cosmological state (accelerating phase) of the Universe. For this reason, Wormhole physicists have borrowed exotic matter sources from Cosmologists [14-23]. In this article, we will give our attention to Chaplygin gas as a supplier of energy to construct a wormhole. We choose Chaplygin gas as a matter source for the following reasons. In 1904, Prof. S. Chaplygin [24] had used matter source that obeys the equation of state as $p = -\frac{A}{\rho}$, $A > 0$ to describe elevating forces on a plain wing in aerodynamics process. This Chaplygin gas model has been supported different classes of observational tests such as supernovae data [25], gravitational lensing [26-27], gamma ray bursts [28], cosmic microwave background radiation [29]. The most important feature of Chaplygin gas is that the squared of sound velocity $v_s^2 = \frac{A}{\rho^2}$ is always positive irrespective of matter density (i.e. this is always positive even in the case of exotic matter). In the present investigation, we shall construct some new classes of wormholes in the framework of Chaplygin gas with equation of state $p = -\frac{A}{\rho}$, $A > 0$. For any matter distribution with spherical symmetry has no contribution at infinity i.e. then it is equivalent to empty space, in other words, the spacetime is a Schwarzschild spacetime. We note that the empty conditions i.e. $p = \rho = 0$ are not feasible for Chaplygin gas equation of state. So, if one wishes to construct Chaplygin wormhole, then one has to match this interior wormhole solution with exterior Schwarzschild solution at some junction interface , say $r = a$. That means Chaplygin wormholes should not obey asymptotically flatness condition.

Basic equations for constructing wormholes:

A static spherically symmetric Lorentzian wormhole can be described by a manifold R^2XS^2 endowed with the general metric in Schwarzschild co-ordinates (t, r, θ, ϕ) as

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega_2^2 \tag{1}$$

Here, $\frac{\nu}{2}$ is redshift function and

$$e^\lambda = [1 - \frac{b(r)}{r}]^{-1} \tag{2}$$

where b(r) is shape function. The radial coordinate runs from r_0 to infinity, where the minimum value r_0 corresponds to the radius of the throat of the wormhole. Since, we are interested to investigate Chaplygin wormhole, so our wormhole spacetime will never asymptotically flat, in other words, we are compelled to consider a 'cut off' of the stress energy tensor at a junction interface, say, at $r = a$.

Using the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, in orthonormal reference frame (with $c = G = 1$) , we obtain the following stress energy scenario,

$$e^{-\lambda}\left[-\frac{1}{r^2}+\frac{\lambda'}{r}\right]+\frac{1}{r^2}=8\pi\rho \tag{3}$$

$$e^{-\lambda}\left[\frac{1}{r^2}+\frac{\gamma'}{r}\right]-\frac{1}{r^2}=8\pi p \tag{4}$$

$$\frac{1}{2}e^{-\lambda}\left[\gamma''+\frac{1}{2}(\gamma')^2-\frac{1}{2}\gamma'\lambda'+\frac{\gamma'-\lambda'}{r}\right]=8\pi p \tag{5}$$

where p(r) = radial pressure = tangential pressure (i.e. pressures are isotropic) and ρ is the matter energy density.

['$\prime$' refers to differentiation with respect to radial coordinate.]

The conservation of stress energy tensor $[T_a^b]_{;b} = 0$ implies,

$$\frac{dp}{dr}=-(p+\rho)\frac{\gamma\prime}{2} \tag{6}$$

Since our source is charaterized by Chaplygin gas, we assume the equation of state as

$$p=-\frac{A}{\rho} \tag{7}$$

Using equations (6) and (7), one can get,

$$\rho^2=\frac{A}{1+Ee^{-\nu}} \tag{8}$$

where E is an integration constant.

Plugging equation (2) in (3) to yield

$$\rho(r)=\frac{b'}{8\pi r^2} \tag{9}$$

The solution of the equation $b(r) = r$ gives the radius of the throat r_0 of the wormhole. Now equation (6) gives the energy density at r_0 as

$$\rho(r_0)=8\pi r_0^2 A \tag{10}$$

Since the shape function b(r) satisfies flaring out condition at the throat (i.e. $b'(r_0) < 1$), then one can have the following restriction as

$$A < \frac{1}{(8\pi r_0^2)^2} \tag{11}$$

Also violation of NEC, $p + \rho < 0$ implies

$$\rho < \sqrt{A} \tag{12}$$

for all $r \epsilon [r_0, a]$.

Toy models of wormholes:

Now we consider several toy models of wormholes.

Specialization one:

Consider the specific form of redshift function as

$$\nu = 2 \ln E(1 + \frac{1}{r^3}) \tag{13}$$

where constant E is the same as in equation(8).

[This function is asymptotically well behaved and always non zero finite for all $r > 0$. So the choice is justified.]

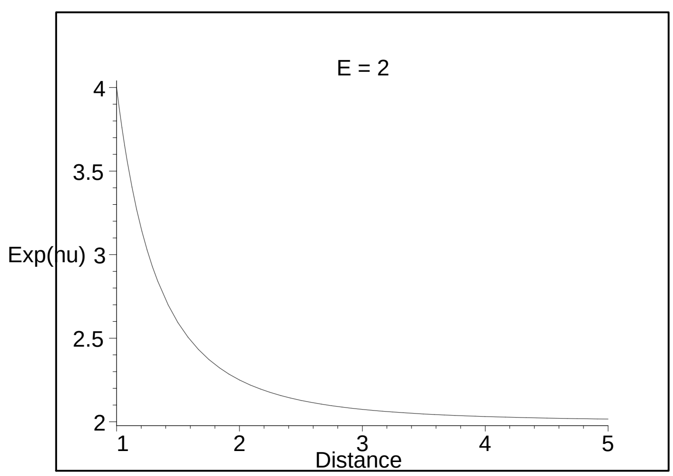

Figure 1: The variation of redshift function with respect to r

For the above redshift function (12), we get an expression for ρ as

$$\rho = \left[\frac{A(r^3+1)}{2r^3+1} \right]^{\frac{1}{2}} \tag{14}$$

By using (9), one can obtain the shape function as

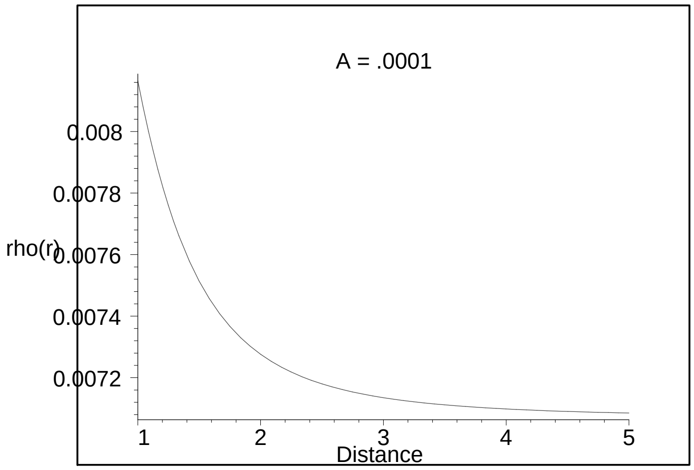

Figure 2: The variation of ρ with respect to r

$$b(r) = \frac{4\pi\sqrt{A}}{3}[\sqrt{2r^6+3r^3+1}] + \frac{2\pi\sqrt{A}}{3}\ln[\sqrt{2r^6+3r^3+1}+2r^3+\frac{3}{2}] \tag{15}$$

We see that $\frac{b(r)}{r}$ does not tend to zero as $r \to \infty$ i.e. the spacetime is not asymptotically

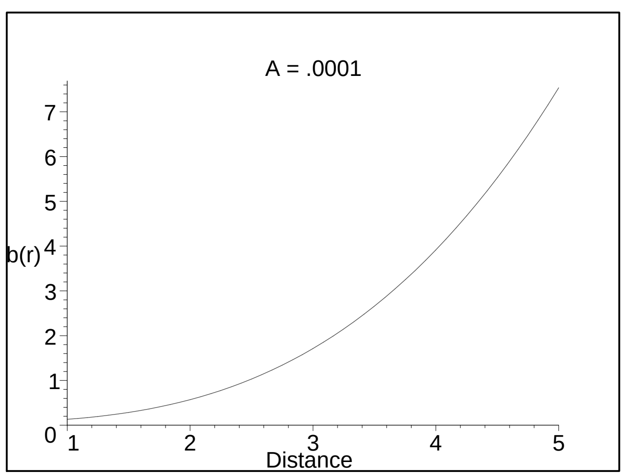

Figure 3: The variation of shape function with respect to r

flat as expected for any Chaplygin wormhole spacetime. Here the throat occurs at $r = r_0$ for which $b(r_0) = r_0$ i.e. $r_0 = \frac{4\pi\sqrt{A}}{3}[\sqrt{2r_0^6+3r_0^3+1}] + \frac{2\pi\sqrt{A}}{3}\ln[\sqrt{2r_0^6+3r_0^3+1}+2r_0^3+\frac{3}{2}]$. For the suitable choice of the parameter, the graph of the function $G(r) \equiv b(r) - r$ indicates the point r_0, where G(r) cuts the 'r' axis (see fig. 4). From the graph, one can also note that when $r > r_0$, $G(r) < 0$ i.e. $b(r) < r$. This implies $\frac{b(r)}{r} < 1$ when $r > r_0$.

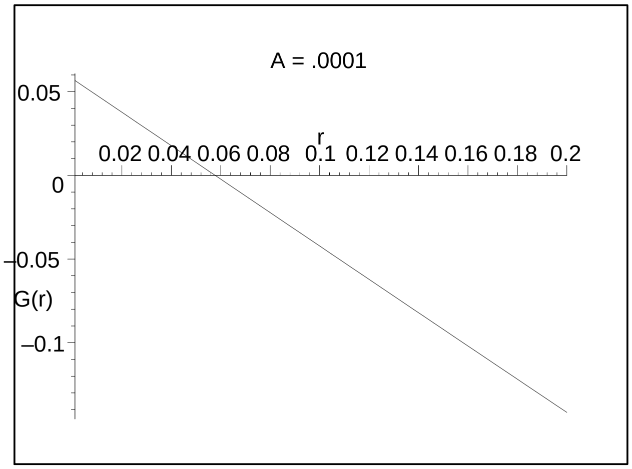

Figure 4: Throat occurs where G(r) cuts r axis

Now we match the interior wormhole metric to the exterior Schwarzschild metric . To match the interior to the exterior, we impose the continuity of the metric coefficients, $g_{\mu\nu}$, across a surface, S , i.e. $g_{\mu\nu(int)}|_S = g_{\mu\nu(ext)}|_S$.

[This condition is not sufficient to different space times. However, for space times with a good deal of symmetry (here, spherical symmetry), one can use directly the field equations to match [30-31]]

The wormhole metric is continuous from the throat, $r = r_0$ to a finite distance $r = a$. Now we impose the continuity of g_{tt} and g_{rr},

$g_{tt(int)}|_S = g_{tt(ext)}|_S$

$g_{rr(int)}|_S = g_{rr(ext)}|_S$

at $r = a$ [i.e. on the surface S] since $g_{\theta\theta}$ and $g_{\phi\phi}$ are already continuous. The continuity of the metric then gives generally

${e^{\nu}}_{int}(a) = {e^{\nu}}_{ext}(a)$ and $g_{rr(int)}(a) = g_{rr(ext)}(a)$.

Hence one can find

$$e^{\nu} = (1 - \frac{2GM}{a}) \tag{16}$$

and $1 - \frac{b(a)}{a} = (1 - \frac{2GM}{a})$ i.e.

$$b(a) = 2GM \tag{17}$$

Equation (16) implies

$a^3(E-1)+2GMa^2+E=0$

Thus matching occurs at $a=S+T-\frac{a_1}{3}$,

where $S=[R+\sqrt{Q^3+R^2}]^{\frac{1}{3}}$ and $T=[R-\sqrt{Q^3+R^2}]^{\frac{1}{3}}$, $Q=\frac{-a_1^2}{9}, R=\frac{-27a_3-2a_1^3}{54}$

with $a_1=\frac{2GM}{E-1}$, $a_3=\frac{E}{E-1}$.

One should note that the above equations (16) and (17) are consistent equations if the solution of $'a'$ (obtained from (16)) should satisfy the equation (17). Thus one can see that the arbitrary constants follow the constraint equation as

$2GM=\frac{4\pi\sqrt{A}}{3}[\sqrt{2(S+T-\frac{a_1}{3})^6+3(S+T-\frac{a_1}{3})^3+1}]$
$+\frac{2\pi\sqrt{A}}{3}\ln[\sqrt{2(S+T-\frac{a_1}{3})^6+3(S+T-\frac{a_1}{3})^3+1}+2(S+T-\frac{a_1}{3})^3+\frac{3}{2}]$.

Now we investigate the total amount of averaged null energy condition (ANEC) violating matter. According to Visser et al[32] this can be quantified by the integral

$$I=\oint(p+\rho)dV=2\int_{r_0}^{\infty}(p+\rho)4\pi r^2dr \tag{18}$$

[$dV=r^2\sin\theta drd\theta d\phi$, factor two comes from including both wormhole mouths]

We consider the wormhole field deviates from the throat out to a radius $'a'$. Thus we obtain the total amount ANEC violating matter as

$I_{total}=-\frac{4\pi\sqrt{A}}{3}\frac{\sqrt{2a^6+3a^3+1}}{2}+\frac{2\pi\sqrt{A}}{\sqrt{2}}\ln[2\sqrt{2}\sqrt{2a^6+3a^3+1}+4a^3+3]$
$+\frac{4\pi\sqrt{A}}{3}\frac{\sqrt{2r_0^6+3r_0^3+1}}{2}-\frac{2\pi\sqrt{A}}{\sqrt{2}}\ln[2\sqrt{2}\sqrt{2r_0^6+3r_0^3+1}+4r_0^3+3]$

This implies that the total amount of ANEC violating matter depends on several parameters, namely, a, A, G, M, E, r_0. If we kept fixed the parameters A, G, M, E, r_0, then the parameter $'a'$ plays significant role to reducing total amount of ANEC violating matter. Thus total amount of ANEC violating matter can be made small by taking suitable position, where interior wormhole metric will match with exterior Schwarzschild metric. This would be zero if one takes $a\rightarrow r_0$.

Specialization two:

Consider the specific form of shape function as

$$b(r) = d \tanh Cr \tag{19}$$

where d and C (> 0) are arbitrary constants.

We will now verify the above particular choice of the shape function would represent wormhole structure. One can easily that $\frac{b(r)}{r} \to 0$ as $r \to \infty$ and throat occurs at r_0 for which $d \tanh Cr_0 = r_0$. The graph indicates the points r_0 where $G(r) \equiv b(r) = r$ cuts the r axis (see fig. 6). Also from the graph, one can note that when $r > r_0$, $G(r) < 0$ i.e. $\frac{b(r)}{r} < 1$ when $r > r_0$.

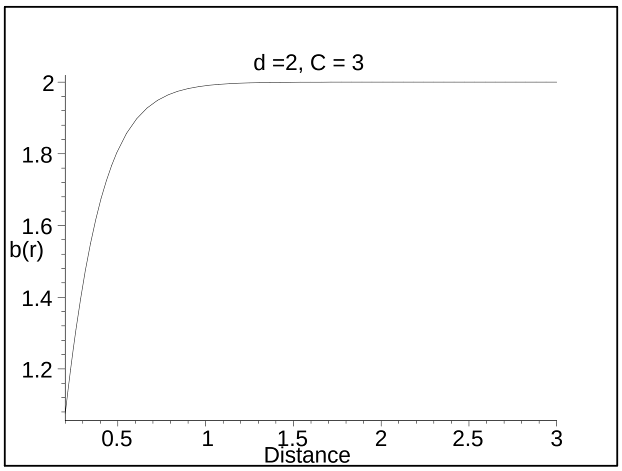

Figure 5: Diagram of the shape function of the wormhole

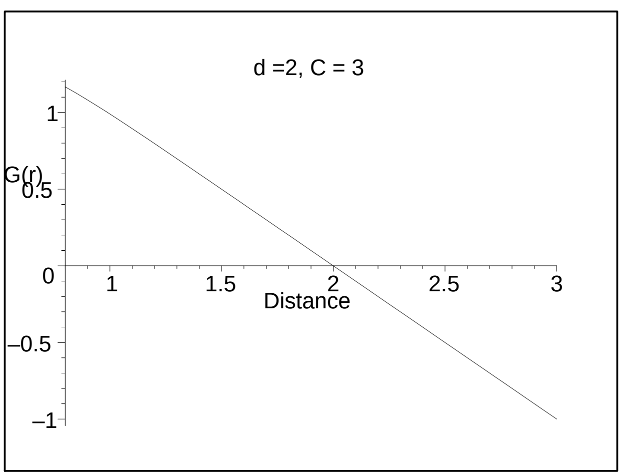

Figure 6: Throat occurs where G(r) cuts r axis

For this shape function, the energy density and redshift function will take the following forms as

$$\rho = \frac{dC}{8\pi r^2 \cosh^2(Cr)} \tag{20}$$

$$e^\nu = \frac{Ed^2C^2}{64A\pi^2 r^4 \cosh^4(Cr) - d^2A^2} \tag{21}$$

As the violation of NEC implies $\rho^2 < A$, we see that e^ν is regular in $[r_0, a]$, where $'a'$ is the cut off radius.

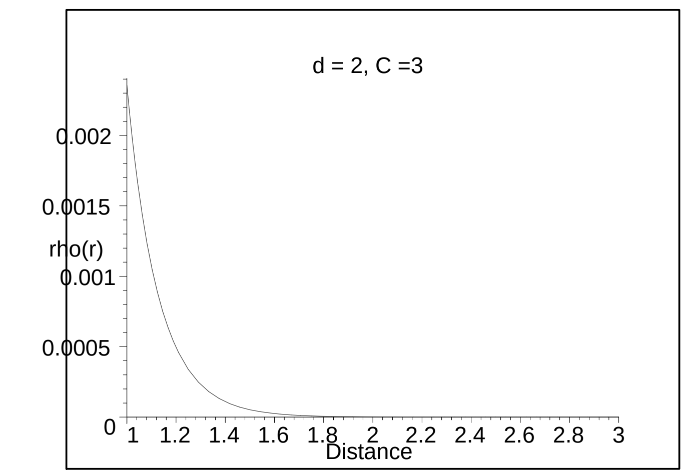

Figure 7: The variation of ρ with respect to r

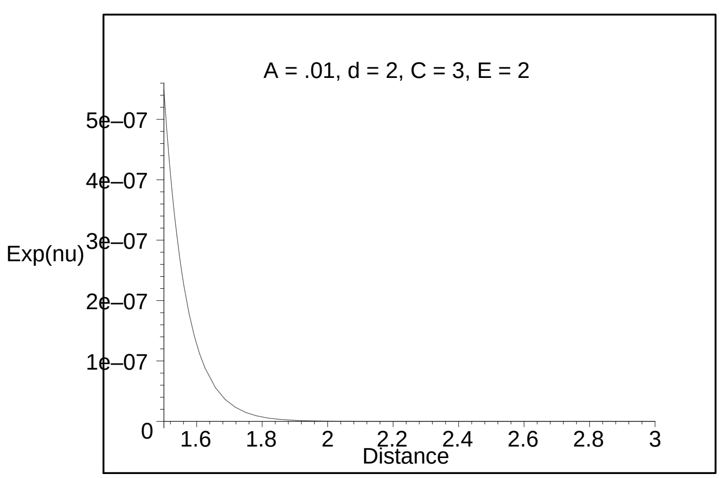

Figure 8: The variation of redshift function with respect to r

Since the wormhole metric is continuous from the throat $r = r_0$ to a finite distance $r = a$, one can use the continuity of the metric coefficients across a surface, as above, i.e, at $'a'$. Here we note that the matching occurs at $'a'$ where $'a'$ satisfies the following equation

$$\alpha a^5 - \beta a^4 - \gamma - \delta = 0 \tag{22}$$

where $\alpha = 64\pi^2 A$, $\beta = 128\pi^2 GMA$, $\gamma = d^2(1-4G^2M^2)^2(A^2+E^2C^2)$ and $\delta = 2GMd^2A^2(1-4G^2M^2)^2$.

For this case, the total amount of ANEC violating matter in the spacetime with a cut off of the strees energy at $'a'$ is given by

$I_{total} = \frac{d}{2}(\tanh Ca - \tanh Cr_0) - \frac{32\pi^2 A}{dC}(\frac{a^5}{10} - \frac{r_0^5}{10}) + \frac{1}{4C}(a^4 \sinh 2Ca - r_0^4 \sinh 2Cr_0) - \frac{1}{2C^2}(a^3 \cosh 2Ca - r_0^3 \cosh 2Cr_0) - \frac{3}{4C^4}(a \cosh 2Ca - r_0 \cosh 2Cr_0) + \frac{3}{4C^3}(a^2 \sinh 2Ca - r_0^2 \sinh 2Cr_0) + \frac{3}{8C^5}(\sinh 2Ca - \sinh 2Cr_0)$

In this case, also, if one treats C, r_0 , d, A are fixed quantities, the total amount of ANEC violating matter can be reduced by taking the suitable position where the interior wormhole metric will match with exterior Schwarzschild metric.

Specialization three:

Now we make specific choice for the shape function as

$$b(r) = D(1 - \frac{F}{r})(1 - \frac{B}{r}) \tag{23}$$

where F, B and D(> 0) are arbitrary constants.

We note that $\frac{b(r)}{r} \to 0$ as $r \to \infty$ and throat occurs at r_0 for which $D(1-\frac{F}{r_0})(1-\frac{B}{r_0}) = r_0$. The graph indicates the points r_0 where $G(r) \equiv b(r) = r$ cuts the 'r' axis (see fig. 10). Also from the graph, one can see that when $r > r_0$, $G(r) < 0$ i.e. $\frac{b(r)}{r} < 1$ when $r > r_0$.

For the above shape function, we find the energy density and redshift function as

$$\rho = \frac{D}{8\pi r^4}[(F + B) - \frac{2FB}{r}] \tag{24}$$

$$e^\nu = \frac{ED^2[(F + B) - \frac{2FB}{r}]^2}{A(8\pi r^4)^2 - D^2[(F + B) - \frac{2FB}{r}]^2} \tag{25}$$

As above, the violation of NEC reflects that e^ν is regular in $[r_0, a]$, where $'a'$ is the cut off radius.

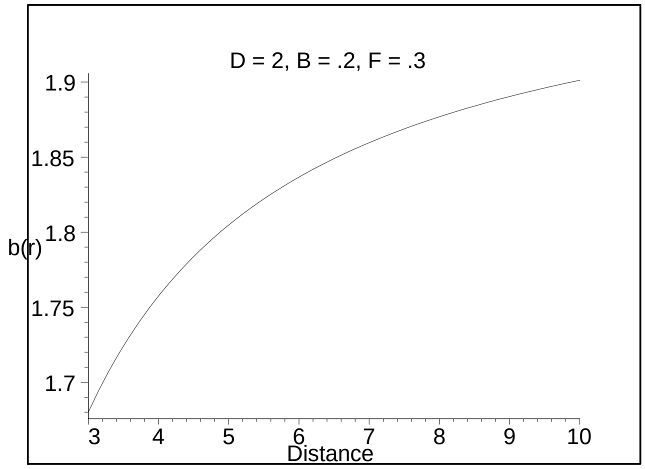

Figure 9: Diagram of the shape function of the wormhole

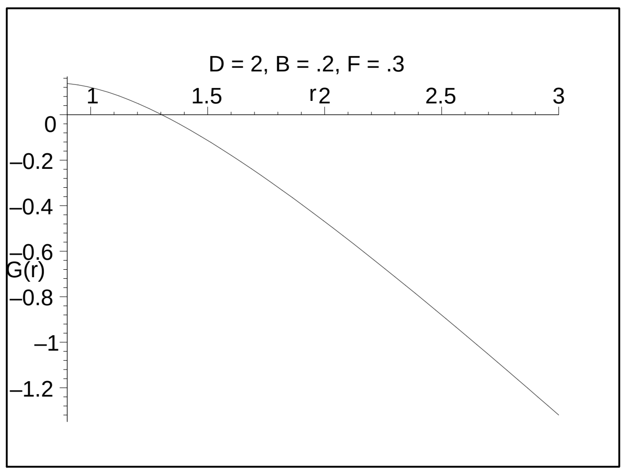

Figure 10: Throat occurs where G(r) cuts r axis

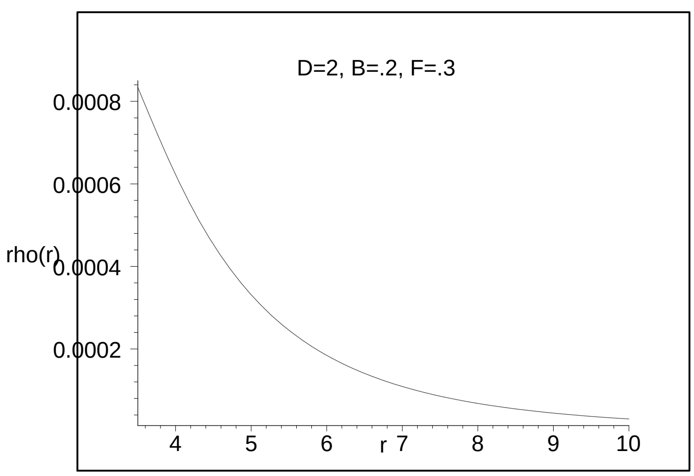

Figure 11: The variation of ρ with respect to r

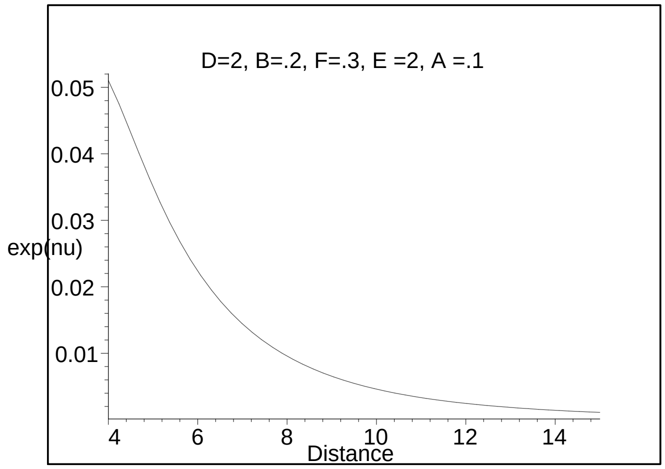

Figure 12: The variation of redshift function with respect to r

Now this interior wormhole metric will match with exterior Schwarzschild metric at $'a'$ where $'a'$ satisfies the following system of consistent equations

$$2GM = D(1-\frac{F}{a})(1-\frac{B}{a}) \tag{26}$$

$$1-\frac{2GM}{a} = \frac{ED^2[(F+B)-\frac{2FB}{a}]^2}{A(8\pi a^4)^2 - D^2[(F+B)-\frac{2FB}{a}]^2} \tag{27}$$

For this specific wormhole model, the total amount ANEC violating matter in the spacetime with a cut off of the stress energy at $'a'$ is

$I_{total} = 8DFB(\frac{1}{a^2}-\frac{1}{r_0^2}) - \frac{D(F+B)}{2}(\frac{1}{a}-\frac{1}{r_0}) - \frac{32\pi^2 A}{7D\alpha}(a^7-r_0^7) - \frac{32\pi^2 A\beta}{6D\alpha^2}(a^6-r_0^6) + \frac{32\pi^2 A\beta^2}{5D\alpha^3}(a^5-r_0^5)\frac{32\pi^2 A\beta^3}{4D\alpha^4}(a^4-r_0^4) + \frac{32\pi^2 A\beta^4}{3D\alpha^8}[(\alpha a+\beta)^3-(\alpha r_0+\beta)^3] - \frac{96\pi^2 A\beta^5}{2D\alpha^8}[(\alpha a+\beta)^2-(\alpha r_0+\beta)^2]\frac{96\pi^2 A\beta^6}{D\alpha^8}[(\alpha a+\beta)-(\alpha r_0+\beta)] - \frac{32\pi^2 A\beta^7}{D\alpha^8}[\ln(\alpha a+\beta)-\ln(\alpha r_0+\beta)^2]$, where $\alpha = F+B$, $\beta = -2FB$.

Similar to previous cases, if one treats C, r_0 , d, A are fixed quantities, the total amount of ANEC violating matter can be reduced by taking the suitable position where the interior wormhole metric will match with exterior Schwarzschild metric.

Final Remarks:

We have investigated how Chaplygin gas can fuel to construct a wormhole. Since General Theory of Gravity admits wormhole solution with the violation of NEC, so Scientists are trying to find matter source that does not obey NEC. We give several specific toy models of wormholes within the framework of Chaplygin gas. Since Chaplygin wormholes are not asymptotically flat, so we have matched our interior wormhole solution with exterior Schwarzschild solution at some junction interface, say, $r = a$ [Actually, if the metric coefficients are not differentiable and affine connections are not continuous at the junction then one has to use the second fundamental forms associated with the two sides of the junction surface]. Though we have assumed several specific forms of either redshift function or shape functions, but in all the cases, one verifies the absence of event horizon. We also quantify the total amount of ANEC violating matter for all models. One can note that the total amount of ANEC violating matter depends on several arbitrary parameters including the position of the matching surface. This position plays crucial role of reducing matter. One can see that this would be infinitesimal small if one takes $a \to r_0$. Our models reveal the fact that one may construct wormholes with arbitrary small amount of matter which is characterized by Chaplygin equation of state. Finally we note that the asymptotic wormhole mass (defined by $M = \lim_{r\to\infty} \frac{1}{2} b(r)$) does not exist for first model but for the second and third models do exist and take the values 'd' and 'D' respectively. In spite of these wormholes are supported by the exotic matter characterized by Chaplygin equation of state, but asymptotic mass is positive. This implies for an observer sitting at large distance could not distinguish the gravitational nature between Wormhole and a compact mass 'M'.

Acknowledgments

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